§3 LINE BUNDLES

A line bundle is a complex vector bundle of rank 1 (we are only interested in complex line bundles with typical fibre isomorphic to $\mathbb{C}$).

(3.1) DEFINITION: A LINE BUNDLE over a given manifold M is a manifold L (the TOTAL SPACE) together with a (smooth) map

$$\pi : L \longrightarrow M$$

with the following properties:

1° Every fibre $L_a := \pi^{-1}(a)$, $a \in M$, has the structure of a one dimensional vector space over $\mathbb{C}$.

2° π is LOCALLY TRIVIAL (the total space L locally looks like a product $U \times \mathbb{C}$ with respect to π): For every point $a \in M$ there exists an open neighbourhood and a diffeomorphism

$$\varphi : L_U := \pi^{-1}(U) \to U \times \mathbb{C}$$

with a) the diagramm

$$\begin{tikzcd} L_U \arrow[r] \arrow[d, "\pi|_{\pi^{-1}(U)}"'] & U \times \mathbb{C} \arrow[dl, "pr_1"] \\ U & \end{tikzcd}$$

is commutative, i.e

$$pr_1 \circ \varphi = \pi|_{\pi^{-1}(U)},$$

b) $\varphi_b : L_b \xrightarrow[\varphi|L_b]{} \{b\} \times \mathbb{C} \xrightarrow[pr_2]{} \mathbb{C}$ is a homomorphism (in fact: an isomorphism) of vector spaces over $\mathbb{C}$.

Note that a line bundle is called "Geradenbündel" in german.

A line bundle is TRIVIAL if $L = M \times \mathbb{C}$ with $\pi = pr_1$ and $L_a = \{a\} \times \mathbb{C}$ obtains its vector space structure from the bijection $L_a = \{a\} \times \mathbb{C} \xrightarrow[pr_2]{} \mathbb{C}$.

However, by abuse of language, the line bundles which are isomorphic to a trivial line bundle are also called TRIVIAL (the precise description would be TRIVIALIZABLE). To understand "isomorphic" we have to introduce the notion of a homomorphism of line bundles.

(3.2) DEFINITION: A LINE BUNDLE HOMOMORPHISM between line bundles $L \xrightarrow{\pi} M$ and $L' \xrightarrow{\pi'} M$ over a manifold M is a smooth map $\psi : L \to L'$ such that $\pi = \pi' \circ \psi$ and such that each $\psi_a := \psi|_{L_a} : L_a \to L'_a$ is a (vector space) homomorphism.

In particular, the diagram

$$\begin{tikzcd} L \arrow[rr, "\psi"] \arrow[dr, "\pi"'] & & L' \arrow[dl, "\pi'"] \\ & M & \end{tikzcd}$$

is commutative.

An isomorphism of line bundles is a homomorphism

φ of line bundles which is bijective such that φ^{-1} is also a line bundle homomorphism. Of course, a homomorphism is already an isomorphism if it is bijective.

(3.3) <u>DEFINITION</u>: Given a line bundle $\pi : L \to M$ and an open subset $\mathcal{U} \subset M$ a SECTION in L over $\mathcal{U}$ is a smooth map

$$s : \mathcal{U} \to L$$

satisfying $\pi \circ s = \mathrm{id}_{\mathcal{U}}$.

The set of section is denoted by $\Gamma(\mathcal{U}, L)$. By pointwise addition and scalar multiplication $\Gamma(\mathcal{U}, L)$ is a vector space over $\mathbb{C}$ and an $\mathcal{E}(\mathcal{U})$-module: For $s, t \in \Gamma(\mathcal{U}, L)$ and $f \in \mathcal{E}(\mathcal{U})$ we set

$$(fs + t)(a) := f(a)s(a) + t(a), \quad a \in \mathcal{U},$$

and see that $fs + t \in \Gamma(\mathcal{U}, L)$.

In case of the trivial bundle $L = M \times \mathbb{C}$ the space $\Gamma(\mathcal{U}, L)$ is naturally isomorphic to $\mathcal{E}(\mathcal{U})$. Let $s_1(a) := (a, 1)$ be the 1-section, $s_1 \in \Gamma(\mathcal{U}, L)$. For each $f \in \mathcal{E}(\mathcal{U})$ we have

$$fs_1(a) = f(a)s_1(a) = f(a)(a, 1) = (a, f(a)), \quad a \in \mathcal{U},$$

hence, the map

$$\mathcal{E}(\mathcal{U}) \longrightarrow \Gamma(\mathcal{U}, M \times \mathbb{C}), \quad f \longmapsto fs_1,$$

is an $\mathcal{E}(U)$-module isomorphism.

(3.4) <u>PROPOSITION:</u> The line bundle $\pi : L \to M$ is trivial (-izable) if and only there exists a global nowhere vanishing section of L, i.e a section $s \in \Gamma(M,L)$ with $s(a) \neq 0$ for all $a \in M$.

☐ <u>Proof</u>. Let s be such a section. It is enough to show that

$$\varphi : M \times \mathbb{C} \longrightarrow L \,, \quad (a,\lambda) \longmapsto \lambda s(a) \,, \text{ for } (a,\lambda) \in M \times \mathbb{C},$$

is a diffeomorphism. Of course, φ is smooth and $\pi \circ \varphi\,(a,\lambda) = \pi(\lambda s(a)) = a$, i.e. $\pi \circ \varphi = pr_1$. And for each $a \in M$

$$\varphi_a = \varphi|_{\{a\} \times \mathbb{C}} \to L_a \,, (a,\lambda) \mapsto \lambda s(a),$$

is an isomorphism of vector spaces. ☐

The fact that for trivial line bundles $L \cong M \times \mathbb{C}$ there is a natural isomorphism $\Gamma(M,L) \cong \mathcal{E}(M)$ is one way of interpreting $\Gamma(M,L)$ as a generalization of the algebra $\mathcal{E}(M)$ of functions. Another such interpretation will be given after the next step in describing line bundles.

The condition 2° in Definition (3.1) yields an open cover $(U_j)_{j \in I}$ of M with trivializations

$$\varphi_j : L_{U_j} \to U_j \times \mathbb{C},$$

that is, φ_j is a diffeomorphism with $\pi \circ \varphi_j = pr_1$ and $\varphi_a = \varphi|_{L_a} : L_a \to \{a\} \times \mathbb{C}$ is an isomorphism.

For each $j \in I$ one obtains a section $s_j \in \Gamma(U_j, L)$ by

$$s_j(a) := \varphi_j^{-1}(a, 1), \quad a \in U_j,$$

with the property

$$\varphi_j\,(z s_j(a)) = (a, z) \quad \text{or} \quad \varphi_j^{-1}(a, z) = z s_j(a)$$

for $(a,z) \in U_j \times \mathbb{C}$. Hence, on $U_j \cap U_k$ the following condition holds

$$s_j = g_{kj}\, s_k \quad , \quad j, k \in I$$

where the "transition functions" ("Übergangsfunktionen") $g_{kj} : U_j \cap U_k \to \mathbb{C}^*$ are defined by

$$\text{"}g_{kj} := \frac{s_j}{s_k}\text{"} \qquad j, k \in I.$$

More precisely, let $U_{jk} := U_j \cap U_k \neq 0$. The composition

$$\varphi_k \circ \varphi_j^{-1} : U_{jk} \times \mathbb{C} \to L_{U_{jk}} \to U_{jk} \times \mathbb{C}$$

acts as $(a, z) \longmapsto (a, g_{kj}(a).z)$, since

$$(\varphi_k \circ \varphi_j^{-1})_a = \varphi_k \circ \varphi_j^{-1}|_{\{a\} \times \mathbb{C}} : \{a\} \times \mathbb{C} \to \{a\} \times \mathbb{C}$$

is an isomorphism of one dimensional complex vector spaces (by definition) an hence given by a non-zero complex number $g_{kj}(a) \in \mathbb{C}^\times$.

The transition functions $(g_{jk})_{j,k\in N}$, $g_{jk}\in\mathcal{E}(U_{jk})$, satisfy the following COCYCLE condition ("Kozyklus-Bedingung")

$$
[C]\qquad
\begin{array}{lll}
g_{jj} = 1 & & \text{on } U_j \\
g_{jk}\,g_{kj} = 1 & & \text{on } U_{jk} = U_j \cap U_k \neq \emptyset \\
g_{jk}\,g_{kl}\,g_{lj} = 1 & & \text{on } U_{jkl} = U_j \cap U_k \cap U_l \neq \emptyset
\end{array}
$$

The transition functions describe the sections in the following way. For each $s \in \Gamma(M,L)$ we get

$$f_j := pr_2 \circ \varphi_j \circ s|_{U_j} \qquad \text{cf.} \qquad
\begin{tikzcd}
L_{U_j} \arrow[r, "\varphi_j", "\sim"'] \arrow[d, "\pi"] & U_j \times \mathbb{C} \arrow[r, "pr_2"] & \mathbb{C} \\
U_j \arrow[u, bend left, "s"] \arrow[urr, dashed, "f_j"'] & &
\end{tikzcd}$$

satisfying

$$s|_{U_j} = f_j s_j :$$

$$s(a) = \varphi_j^{-1} \circ \varphi_j(s(a)) = \varphi_j^{-1}(a, pr_2 \circ \varphi_j \circ s(a)) = \varphi_j^{-1}(a, f_j(a))$$

$$= f_j(a)\,\varphi_j^{-1}(a,1) = f_j(a)\,s_j(a).$$

On $U_{jk} \neq \emptyset$ we obtain

$$s|_{U_{jk}} = f_k s_k|_{U_{jk}} = f_j s_j|_{U_{jk}} = f_j g_{kj} s_k|_{U_{jk}}$$

arriving at the "section condition":

$$[S] \qquad f_k = g_{kj} f_j \qquad \text{on } U_{jk}.$$

(3.5) <u>PROPOSITION:</u> 1° $s \in \Gamma(M,L)$ defines a collection $(f_j)_{j\in I}$, $f_j \in \mathcal{E}(U_j)$, with [S].

[4.11.09]

2° Conversely, every collection $(f_j)_{j\in I}$, $f_j \in \mathcal{E}(U_j)$, satisfying [S] yields a global section $s \in \Gamma(M,L)$ in L with $s|_{U_j} = f_j s_j$, $j \in I$.

□ Proof. 1° has just been shown. The data (f_j) in 2° have the property $f_j s_j|_{U_{jk}} = f_k s_k|_{U_{jk}}$ by [S] and thus define a global section s by

$$s(a) := f_j s_j(a) \quad , a \in U_j,$$

with $s|_{U_j} = f_j s_j$! □

Note that the result of the proposition (3.5) also gives an interpretation of $\Gamma(M,L)$ as generalized functions. This generalization is adapted to our problem of not having a global potential for a given symplectic form: The U_j can alway be chosen in such a way that there exist $\alpha_j \in \Omega^1(U_j)$ with $d\alpha_j = \omega|_{U_j}$.

But are there non-trivial line bundles at all?

(3.6) EXAMPLE: The "tautological bundle" on $\mathbb{P}_n(\mathbb{C})$.
Let $\mathbb{P}_1(\mathbb{C})$ be the Riemann sphere resp. the 1-dimensional projective space. $\mathbb{P}_1 = \mathbb{P}_1(\mathbb{C})$ is the space of lines in $\mathbb{C}^2$ through $0 \in \mathbb{C}^2$, i.e. the space of one dimensional linear subspaces of $\mathbb{C}^2$. We have a natural projection

$$\varkappa : \mathbb{C}^2 \setminus \{0\} \longrightarrow \mathbb{P}_1(\mathbb{C})$$

mapping each line $l \subset \mathbb{C}^2 \setminus \{0\}$ to its corresponding points in $\mathbb{P}_1(\mathbb{C})$. Hence $\varkappa$ is the projection with respect to the following equivalence relation in $\mathbb{C}^2 \setminus \{0\}$:

$$z \sim w \iff \exists \lambda \in \mathbb{C} : z = \lambda w .$$

The points in $\mathbb{P}_1(\mathbb{C})$ are described by the so called HOMOGENEOUS COORDINATES coming from $\mathbb{C}^2$: For $z = (z_0, z_1) \in \mathbb{C}^2 \setminus \{0\}$, we set

$$\pi(z) = (z_0 : z_1) \quad \left(= [z] \text{ the equivalence class of } z \right)$$

with the understanding of $\pi(\lambda z) = \pi(z)$, i.e.

$$(z_0 : z_1) = (\lambda z_0 : \lambda z_1) \quad , \text{ if } \lambda \in \mathbb{C}^\times = \mathbb{C} \setminus \{0\}.$$

$\mathbb{P}_1(\mathbb{C})$ gets its topological and complex structure by the projection π: $\mathbb{P}_1(\mathbb{C})$ is the quotient $\mathbb{P}_1(\mathbb{C}) = \mathbb{C}^2 \setminus \{0\} / \sim$ as a topological space and as a complex manifold. Hence, $U \subset \mathbb{P}_1(\mathbb{C})$ is open iff $\pi^{-1}(U)$ is open and a map $f : U \to \mathbb{C}$ is holomorphic if $f \circ \pi : \pi^{-1}(U) \to \mathbb{C}$ is holomorphic.

$\mathbb{P}_1(\mathbb{C})$ can be covered by two holomorphic charts $\psi_j : U_j \to \mathbb{C}$, where $U_j := \{(z_0 : z_1) : z_j \neq 0\}$ and

$$\psi_0 : U_0 \to \mathbb{C}, \quad (1 : z) \longmapsto z, \quad z \in \mathbb{C},$$

and

$$\psi_1 : U_1 \longrightarrow \mathbb{C}, \quad (w : 1) \longmapsto w, \quad w \in \mathbb{C}.$$

On $U_{01} = U_0 \cap U_1 \neq \emptyset$ we have

$$\psi_0 \circ \psi_1^{-1}(w) = \psi_0(w : 1) = \psi_0\left(1 : \frac{1}{w}\right) = \frac{1}{w} .$$

This is a holomorphic function on $\mathbb{C}^\times = \psi_1(U_{01})$ with a holomorphic inverse. Therefore, the holomorphic (and also the $\mathcal{C}^\infty$-)structure is given also by these two charts.

U_0 could be understood as the plane $\mathbb{C}$ with $(1:0)$ as 0, and $U_1 \subset \mathbb{P}_1(\mathbb{C})$ adds only the point "$\infty = (0:1)$" to U_0, thus obtaining the sphere $\mathbb{S}^2 \overset{"}{=} \mathbb{C} \cup \{\infty\}$".

Now, to come to the tautological bundle we with the product $\mathbb{P}_1 \times \mathbb{C}^2$ which is a trivial holomorphic rank 2 vector bundle. Define

$$T := \left\{ (a,w) \in \mathbb{P}_1 \times \mathbb{C}^2 \;\middle|\; \exists \lambda \in \mathbb{C} \;:\; w = (\lambda a_0, \lambda a_1) \text{ if } a = (a_0 : a_1) \right\}$$

$$= \left\{ (a,w) \in \mathbb{P}_1 \times \mathbb{C}^2 \;\middle|\; w = 0 \text{ or } \gamma(w) = a \right\}$$

and $\pi : T \to \mathbb{P}_1$, $\pi(a,w) := a$. $T \subset \mathbb{P}_1 \times \mathbb{C}^2$ is a complex submanifold of (complex) dimension 2.
For each $a \in \mathbb{P}_1$ the fibre $T_a = \pi^{-1}(a)$ is

$$T_a = \{ (a,w) \mid w = 0 \text{ or } \gamma(w) = a \} = \{a\} \times (a \cup \{0\})$$

Hence T_a is precisely the line given by the equivalence class $a \in \mathbb{P}_1(\mathbb{C})$. This is the reason why T is called the tautological bundle. T_a obtains the natural structure of a complex vector space by using this equality $T_a = \{a\} \times (a \cup \{0\}) \cong (a \cup \{0\}) \subset \mathbb{C}^2$.

Moreover, we have for $j = 0,1$

$$T_{U_j} = \pi^{-1}(U_j) \longrightarrow U_j \times \mathbb{C} \quad , \quad (a,w) \longmapsto (a, w_j) =: \varphi_j(a,w).$$

$(w = (w_0, w_1) \in \mathbb{C}^2)$ the diffeomorphisms $\varphi_j : T_{U_j} \to U_j \times \mathbb{C}$

which are local triviliazations of T with respect to the coordinate neighbourhoods $\mathcal{U}_0, \mathcal{U}_1$.

Because of $\varphi_0^{-1}(a,z) = (a, (z, \frac{a_1}{a_0}z))$ for $a \in \mathcal{U}_0$ (i.e. $a=(a_0 : a_1)$ with $a_0 \neq 0$) the transition

$$\mathcal{U}_{01} \times \mathbb{C} \xrightarrow{\varphi_0^{-1}} T_{\mathcal{U}_{01}} \xrightarrow{\varphi_1} \mathcal{U}_{01} \times \mathbb{C}$$

is

$$\varphi_1 \circ \varphi_0^{-1} : (a,z) = \varphi_1 \left(a, \left(z, \frac{a_1}{a_0} z\right)\right) = \left(a, \frac{a_1}{a_0} z\right),$$

$z \in \mathbb{C}$ and $a \in \mathcal{U}_{01} = \mathcal{U}_0 \cap \mathcal{U}_1$. Hence the corresponding $g_{10} : \mathcal{U}_{01} \to \mathbb{C}^{\times}$ (with $\varphi_1 \circ \varphi_0^{-1}(a,z) = (a, g_{10}(a) z)$ is

$$g_{10}(a) = \frac{a_1}{a_0}, \quad a = (a_0 : a_1) \in \mathcal{U}_{01}.$$

Analoguously, $g_{01}(a) = \frac{a_0}{a_1}$ and we see, that [C] is satisfied.

It is rather evident, that $T \to \mathbb{P}_1(\mathbb{C})$ is also the tangent bundle of $\mathbb{S}^2 = \mathbb{P}_1(\mathbb{C})$ if we consider the $\mathcal{C}^\infty$-structure on $\mathbb{P}_1(\mathbb{C})$.[*] Now $T = T\mathbb{S}^2 \to \mathbb{S}^2$ is known to be non-trivial according to the "Satz vom Igel": There are no vector fields on the sphere without any zero: If $X : \mathbb{S}^2 \to T\mathbb{S}^2$ is a smooth section, there always exits $a \in \mathbb{S}^2$ with $X(a) = 0$.

That $T \to \mathbb{P}_1(\mathbb{C})$ has no holomorphic trivialization

$$\varphi : T \xrightarrow{\sim} \mathbb{P}_1(\mathbb{C}) \times \mathbb{C}$$

(i.e. φ isomorphism and holomorphic) can be seen

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* Übung: Check in detail!

by the fact that $T \to M$ only admits the zero section as a holomorphic section: $\Gamma_{hol}(\mathbb{P}_1, T) = \{0\}$. Indeed, a holomorphic $s : \mathbb{P}_1 \to T$ with $\pi \circ s = id_{\mathbb{P}_1}$ is given by holomorphic functions $f_j : U_j \to \mathbb{C}$ with

$$f_0(z_0 : z_1) = \frac{z_0}{z_1} f_1(z_0 : z_1) \quad , \quad z_j \in \mathbb{C}^\times ,$$

according to the calculation above. The two functions

$$g_j : \gamma^{-1}(U_j) \to \mathbb{C} \, , \quad g_j(\xi) = \xi_j^{-1} f_j(\gamma(\xi)), \; \xi \in \gamma^{-1}(U_j)$$

are well-defined and they agree on $\gamma^{-1}(U_{01})$:

$$g_0(\xi) = \xi_0^{-1} f_0(\gamma(\xi)) = \xi_0^{-1} \frac{\xi_0}{\xi_1} f_1(\gamma(\xi)) = g_1(\xi) .$$

Hence, s determines a holomorphic $g : \mathbb{C}^2 \setminus \{0\} \to \mathbb{C}$ satisfying

$$g(\lambda\xi) = \lambda^{-1} g(\xi) \quad \text{for} \quad (\xi, \lambda) \in (\mathbb{C}^2 \setminus \{0\}) \times \mathbb{C}^\times .$$

But such a holomorphic g is the zero function which in turn implies that s has to be zero. (See (3.8) and (3.9) for more general results.)

The whole consideration of the example can be generalized to the n-dimensional projective space $\mathbb{P}_n(\mathbb{C})$. We describe this in a sequence of formulas and statements:

$$\mathbb{P}_n(\mathbb{C}) = \mathbb{P}_n := \mathbb{C}^{n+1} \setminus \{0\} / \sim \quad \text{with respect to}$$

$$w \sim z \quad (\text{for } z, w \in \mathbb{C}^{n+1} \setminus \{0\}) \; :\Longleftrightarrow \; \exists \lambda \in \mathbb{C}^\times : w = \lambda z \, ,$$

and with the projection

$$\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}_n(\mathbb{C}), \quad \pi(z) = [z] = (z_0 : z_1 : \ldots : z_n)$$

$$z = (z_0, z_1, \ldots z_n) \in \mathbb{C}^{n+1} \setminus \{0\}.$$

$\mathbb{P}_n(\mathbb{C})$ obtains its topological, differentiable and complex structure as the quotient with respect to $\sim$. A suitable holomorphic atlas is given by the following charts:

$$U_j := \{ (z_0 : z_1 : \ldots : z_n) \mid z_j \neq 1 \}$$

$$\psi_j : U_j \to \mathbb{C}^n, \quad (z_0 : z_1 : \ldots : z_n) \longmapsto \frac{1}{z_j}(z_0, z_1, \ldots, \hat{z}_j, \ldots z_n)$$

with biholomorphic $\psi_j \circ \psi_k^{-1} : \psi_k(U_{jk}) \to \psi_j(U_{jk})$. [*]

$\mathbb{P}_n \times \mathbb{C}^{n+1}$ is a trivial holomorphic vector bundle.

$$T := \{ (z, \xi) \in \mathbb{P}_n \times \mathbb{C}^{n+1} \mid \exists \lambda \in \mathbb{C} : \ \xi = \lambda (z_0, z_1, \ldots z_n) \} \text{ is}$$

a complex submanifold $T \subset \mathbb{P}_n \times \mathbb{C}^{n+1}$ of complex dimension $n+1$ and it is the total space of the holomorphic line bundle

$$T \xrightarrow{\pi} \mathbb{P}_n, \quad (z, \xi) \longmapsto z,$$

which is again not trivializable [*], neither holomorphically (see below (3.10)) nor as a differentiable complex line bundle.

For the local trivializations one takes

$$\varphi_j : T_{U_j} \to U_j \times \mathbb{C}, \quad (z, \xi) \longmapsto (z, \xi_j) = \varphi_j(z, \xi),$$

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* Übung: Prove all these statements.

with the corresponding transition functions

$$g_{jk}(z) = \frac{z_j}{z_k} \quad , \quad z = (z_0 : z_1 : \dots : z_n) \in U_{jk} .$$

A generalization of the tautological line bundle on $\mathbb{P}_n(\mathbb{C})$, in particular on $\mathbb{P}_1(\mathbb{C})$, is given by replacing the transition functions

$$g_{jk}(z_0 : \dots : z_n) = \frac{z_j}{z_k} \quad \text{on } U_{jk} = \{ z_k \neq 0 \neq z_j \}$$

by

$$g_{jk}^{(m)}(z_0 : \dots z_n) = \left(\frac{z_k}{z_j} \right)^m \quad \text{''} \quad \text{''} \quad ,$$

where $m \in \mathbb{Z}$. We prove below that such $g_{jk}^{(m)}$ also define a complex line bundle which we denote by $H(m)$. In particular, $T = H(-1)$. $H = H(1)$ is called the HYPERPLANE bundle.[*]

In order to understand the holomorphic line bundles $H(m)$, $m \in \mathbb{Z}$, over $\mathbb{P}_n$ we use the one-to-one correspondence between (isomorphism classes of) line bundles over a manifold M and (equivalence classes of) cocycles (g_{ij}). As a first step:

(3.7) PROPOSITION: Let $(U_j)_{j \in I}$ be an open cover of the manifold M, and let $g_{jk} \in \mathcal{E}(U_{jk}, \mathbb{C})$ be functions forming a cocycle $(g_{jk})_{j,k \in I}$, i.e. such that (g_{jk}) satisfies [C]. Then the data $(M, (U_j), (g_{jk}))$ induce a complex line bundle

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* Übung: Why?

$L \xrightarrow[\pi]{} M$ with local trivializations

$$\varphi_j : L_{U_j} \to U_j \times \mathbb{C}$$

such that for $a \in U_{jk}$ and $z \in \mathbb{C}$:

$$\varphi_j^{-1} \circ \varphi_k (a,z) = (a, g_{jk}(a) z).$$

□ Proof: On the disjoint union $\dot{\bigcup}_{j \in I} U_j \times \mathbb{C} =: R$, which is a $(\dim M + 1)$-dimensional manifold we consider the equivalence relation

$$(a_j, z_j) \sim (a_k, z_k) \quad :\Longleftrightarrow \quad a_j = a_k \;\&\; z_j = g_{jk}(a) z_k$$

for $(a_j, z_j) \in U_j \times \mathbb{C}$, $(a_k, z_k) \in U_k \times \mathbb{C}$.

The quotient manifold $L = R/_{\sim}$ exists[*] with the projection $\pi : L \to M$, $[(a,z)] \mapsto a$, as smooth mapping and with fibres $\pi^{-1}(a) = L_a = \{[(a,z)] : z \in \mathbb{C}\} \xrightarrow{\sim} \mathbb{C}$ as one dimensional complex vector spaces. The trivialisations

$$\varphi_j : L_{U_j} \to U_j \times \mathbb{C}, \quad [(a,z)] \mapsto U_j \times \mathbb{C}, \quad (a,z) \in U_j \times \mathbb{C}$$

lead to the transition functions (g_{jk}) with which started. □

In order to determine the sections of our example of the tautological line bundle $T \to \mathbb{P}_n(\mathbb{C})$ and the other line bundles $H(m) \to \mathbb{P}_n(\mathbb{C})$ over $\mathbb{P}_n(\mathbb{C})$ we introduce the space $\mathcal{E}_m(V) \subset \mathcal{E}(V)$ of smooth function on a

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* Übung: Check that L is well-defined with the stated properties.

saturated open subset $V \subset \mathbb{C}^{n+1} \setminus \{0\}$, i.e. $\varkappa^{-1}(\varkappa(V)) = V$:

$$\mathcal{E}_m(V) := \{ g \in \mathcal{E}_m(V, \mathbb{C}) \mid \forall \lambda \in \mathbb{C}^\times \forall z \in V : g(\lambda z) = \lambda^m g(z) \}.$$

Let $\mathcal{U} \subset \mathbb{P}_n(\mathbb{C})$ be open and $V := \varkappa^{-1}(\mathcal{U})$. Then every smooth section $s \in \Gamma(\mathcal{U}, H(m))$ determines a function

$$\tilde{s} = g_s \in \mathcal{E}_m(V)$$

in the following way. With respect to the open cover $(\mathcal{U}_j)_{0 \le j \le n}$ and the transition functions

$$g_{jk}(z) = \left(\frac{z_k}{z_j}\right)^m, \quad z = (z_0 : z_1 : \dots : z_n), \quad z_j \neq 0 \neq z_k$$

the given section s determines $f_j \in \mathcal{E}(\mathcal{U} \cap \mathcal{U}_j)$, $j = 0, \dots n$, with

$$f_j = g_{jk} f_k \quad \text{on } \mathcal{U}_{jk} \cap \mathcal{U},$$

in fact, $s|_{\mathcal{U} \cap \mathcal{U}_j} = f_j s_j$, $s_j(z) = \varphi_j^{-1}(z, 1)$. We define

$$g_j(\xi) := \xi_j{}^m f_j(\varkappa(\xi)), \quad \xi \in \varkappa^{-1}(\mathcal{U} \cap \mathcal{U}_j) = V \cap \varkappa^{-1}(\mathcal{U}_j)$$

For $\xi \in \varkappa^{-1}(\mathcal{U} \cap \mathcal{U}_{jk})$ we get

$$g_j(\xi) = \xi_j^m g_{jk}(\varkappa(\xi)) f_k(\varkappa(\xi)) = \xi_j^m \left(\frac{\xi_k}{\xi_j}\right)^m f_k(\varkappa(\xi)) = g_k(\xi).$$

As a consequence,

$$g_s(\xi) := g_j(\xi) \quad \text{if} \quad \xi \in \varkappa^{-1}(\mathcal{U} \cap \mathcal{U}_j) = V \cap \varkappa^{-1}(\mathcal{U}_j)$$

is a well-defined function $\tilde{s} = g_s \in \mathcal{E}(V)$. Moreover,

$$g_s(\lambda \xi) = \lambda^m \xi_j^m f_j(\varkappa(\xi)) = \lambda^m g_s(\xi), \quad (\xi, \lambda) \in V \cap \varkappa^{-1}(\mathcal{U}_j) \times \mathbb{C}.$$

Hence, $\tilde{s} = g_s \in \mathcal{E}_m(V)$.
Clearly, the map $\sim : \Gamma(\mathcal{U}, H(m)) \to \mathcal{E}_m(V)$ is linear over $\mathbb{C}$ and injective ($\sim$ is even $\mathcal{E}(\mathcal{U})$-linear). We have shown a main part of the following:

(3.8) <u>PROPOSITION:</u> For every open $\mathcal{U} \subset \mathbb{P}_n$ and $V := \bar{\jmath}^{-1}(\mathcal{U})$ there is a natural isomorphism

$$\sim : \Gamma(\mathcal{U}, H(m)) \longrightarrow \mathcal{E}_m(V).$$

☐ Proof. It remains to show that "$\sim$" is surjective.
$g \in \mathcal{E}_m(V)$ and $z \in \mathcal{U} \cap \mathcal{U}_j$ set

$$f_j(z) := \xi_j^{-m} g(\xi) \quad \text{for } \xi \in \bar{\jmath}^{-1}(z).$$

In case of $\xi' = \lambda \xi$, $\lambda \in \mathbb{C}^\times$, we have

$$\xi_j'^{-m} g(\xi') = \lambda^{-m} \xi_j^{m} \lambda^m g(\xi) = \xi_j^{-m} g(\xi).$$

Therefore, $g_j \in \mathcal{E}(\mathcal{U} \cap \mathcal{U}_j)$ is well-defined and satisfies [S]:

$$g_j(z) = \xi_j^{-m} g(\xi) = \left(\frac{\xi_k}{\xi_j}\right)^m \xi_k^{-m} g(\xi) = \left(\frac{z_k}{z_j}\right)^m g_k(z).$$

Consequently, (g_j) defines a section $s \in \Gamma(\mathcal{U}, H(m))$ such that $\tilde{s} = g$. ☐

We immediately deduce:

(3.9) <u>PROPOSITION:</u>

$$\Gamma_{hol}(\mathbb{P}_n(\mathbb{C}), H(m)) \cong \begin{cases} \{0\} & \text{for } m < 0, \\ \mathbb{C}^{(m)}[z_0, \dots z_n] & \text{for } m \geq 0, \end{cases} \quad m \in \mathbb{Z}.$$

where $\mathbb{C}^{(m)}[z_0, \dots z_n]$ denotes the $\mathbb{C}$-vector space of m-homogeneous polynomials in the $n+1$ variables $z_0, z_1, \dots \quad z_n$.

□ Proof. One only has to check that for $m \geq 1$

$$\mathcal{E}_m(\mathbb{C}^{n+1} \setminus \{0\}) \cap \mathcal{O}(\mathbb{C}^{n+1} \setminus \{0\}) = \mathbb{C}^{(m)}[z_0, \dots z_n] \quad ,$$

where $\mathcal{O}(V)$ is the vector space of holomorphic functions on V. Since each $g \in \mathcal{O}(\mathbb{C}^{n+1} \setminus \{0\})$ has a holomorphic extension to all of $\mathbb{C}^{n+1}$ (if $n \geq 1$: there are no isolated singularities for holomorphic functions of $n+1 \geq 2$ variables) it is enough to observe that the m-homogeneous holomorphic function $g : \mathbb{C}^{n+1} \to \mathbb{C}$ are m-homogeneous polynomials. The case $m \leq 0$ is another exercise. □

As an immediate consequence we obtain the following result

(3.10) COROLLARY: 1° The $\mathbb{C}$-vector space $\Gamma_{hol}(\mathbb{P}_n, H(m))$ of holomorphic sections of the holomorpic line bundle $H(m)$ is finite dimensional.

2° All $H(m)$, $m \neq 0$, are nontrivial as holomorphic line bundles.

3° The line bundles $H(m)$ & $H(k)$ for $m, k \geq 0$, $m \neq k$, are not isomorphic as holomorphic line bundles.

To conclude this section we observe that two cocycles (g_{jk}), (h_{jk}) with respect to an open cover $(U_j)_{j \in I}$

on a manifold M can be multiplied to yield another cocycle

$$f_{jk} := g_{jk} h_{jk} \in \mathcal{E}(\mathcal{U}_{jk}).$$

In this way one defines a composition on the set

$$\mathrm{Pic}^\infty(M)$$

of isomorphism classes of complex line bundles on M. The composition turns out to be associative and commutative. The class of the trivial bundle acts as a unit and the inverse of a classe $[L] \in \mathrm{Pic}^\infty(M)$, represented by the cocycle (g_{jk}) for L is given by the class $[L^\vee]$ where $L^\vee$ is defined by the cocycle (g_{jk}^{-1}). $\mathrm{Pic}^\infty(M)$ is a group the so-called PICARD GROUP. Note, that the product

$$(g_{jk}), (h_{jk}) \longmapsto (g_{jk} h_{jk})$$

can also be given as the tensor product

$$L, M \longmapsto L \otimes M ,$$

when L is given by (g_{jk}) and M by (h_{jk}). The group multiplication in $\mathrm{Pic}^\infty(M)$ can thus be described by

$$[L], [M] \longmapsto [L \otimes M]$$

In case of a complex manifold M one introduces the "holomorphic" Picard group which is

$$\mathrm{Pic}(M) = \{[L] : L \text{ a holomorphic line bundle}\}$$

where $[L]$ is the class of holomorphic line bundles which are holomorphically isomorphic to L, and where the multiplication is again given by $(g_{jk}), (h_{jk}) \longmapsto (g_{jk}h_{jk})$, resp. $L, M \mapsto L \otimes M$.

Using the definition and description of our holomorphic line bundles $H(m) \to \mathbb{P}_n$ we see that $H(n) \otimes H(m) = H(n+m)$ and $H(n) \otimes H(-n) = H(0)$.
Hence the $[H(m)]$, $m \in \mathbb{Z}$, form a subgroup of $\mathrm{Pic}^\infty(\mathbb{P}_n)$ and $\mathrm{Pic}(\mathbb{P}_n)$ isomorphic to $\mathbb{Z}$.
Note, that $\mathrm{Pic}(M) \cong \mathbb{Z}$ can be shown, i.e.
$\mathrm{Pic}(\mathbb{P}_n) = \{[H(m)] : m \in \mathbb{Z}\}$.

The Picard group $\mathrm{Pic}^\infty(M)$ can be identified with the 1. Čech cohomology group

$$H^1(M, \mathcal{E}^\times) \cong \mathrm{Pic}^\infty(M)$$

with respect to the sheaf $\mathcal{E}^\times$ of germs of nowhere vanishing smooth functions. Furthermore, $H^1(M, \mathcal{E}^\times)$ identifies with $H^2(M, \mathbb{Z})$ by the connecting homomorphism $H^1(M, \mathcal{E}^\times) \xrightarrow[\delta]{} H^2(M, \mathbb{Z})$ coming from the exact sequence $0 \to \mathbb{Z} \to \mathcal{E} \xrightarrow{e} \mathcal{E}^\times \to 1$.
In the holomorphic case $\mathrm{Pic}(M)$ is $H^1(M, \mathcal{O}^\times)$, where $\mathcal{O}^\times$ is the sheaf of germs of nowhere vanishing holomorphic functions.

[11.11.09]